Circulant Digraphs Integral over Number Fields

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Abstract A number field K is a finite extension of rational number field $\mathbb{Q}$. A circulant digraph integral over K means that all its eigenvalues are algebraic integers of K. In this paper we give the sufficient and necessary condition for circulant digraphs which are integral over a number field K. And we solve the Conjecture3.3 in [XM] and find it is affirmative.

Keywords: circulant graph, integral graph, algebraic integer, number field

1 Introduction and Main Results

Let G be a finite group and S be a subset of G. The Cayley digraph $D = D(G, S)$ of G with respect to S is a directed graph with vertex set G. For $g_1, g_2 \in G$, there is an arc from g_1 to g_2 if and only if $g_2g_1^{-1} \in S$. If G is the cyclic group $\mathbb{Z}/n\mathbb{Z}$ with order n, the Cayley digraph $D = D(\mathbb{Z}/n\mathbb{Z}, S) = D(n, S)$ is called a circulant digraph. Its spectrum consists of the eigenvalues of the adjacent matrix of D, namely, the roots of the characteristic polynomial of D, that is $spec(D(n, S)) = \{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}\}$,

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where $\lambda_r = \sum_{s\in S}\zeta_n^{rs}(1 \le r \le n-1)$ with $\zeta_n = e^{\frac{2\pi i}{n}}$, the primitive $n-$th root of unity(see [Bi], [GR]). It is obvious that $\lambda_r(1 \le r \le n-1)$ are algebraic integers in the cyclotomic field $\mathbb{Q}(\zeta_n)$. A circulant digraph integral over number field K means that all its eigenvalues are algebraic integers of K. The reader is referred to [F], [N] and [W] for terms left undefined and for the general theory of number fields.

Our main purpose in this paper is to give the sufficient and necessary condition for circulant digraphs which are integral over a number field K. And we solve the Conjecture3.3 in [XM] and find it is affirmative. There are a lot of literature studying integral graphs(see e.g., [Ba][BC][EH][HS][LL][S][Z]).

The main results in the present paper are stated as follows.

A. (see Theorem 1.) The circulant directed graph $D(n, S)$ is integral over K if and only if S is a union of some M_i's.

B. (see Corollary 1.) There are at most $2^{r(n,K)}$ basic circulant digraphs integral over K on n vertices.

Using the same notations of [XM], we have the following corollary.

C. (see Corollary 2.) (Conjecture 3.3 in [XM]) Let $D(\mathbb{Z}_n, S)$ be a circulant digraph on $n = 2^t k (t > 2)$ vertices, then $D(\mathbb{Z}_n, S)$ is Gauss integral if and only if S is a union of the M's.

2 Constructing a partition of $\{1, 2, \cdots, n-1\}$

Let L/M be a number field extension. We denote $Gal(L/M)$ the Galois group of L over M and $[L : M]$ the dimension of L as vector space over M. Let K be a number field, i.e., a finite extension of rational number field $\mathbb{Q}$, and $d(n)$ be the set

of all the positive proper divisors of n and for $p \in d(n)$, we write $n = pg$. Denote $F = K \bigcap \mathbb{Q}(\zeta_n)$ and for $p \in d(n)$ define $G_n(p) = \{g|1 \leq g < n, gcd(g,n) = p\}$. It is clear that $G_n(p) = pG_g(1)$. Notice that F is an abelian field and $D(n,S)$ is integral over K if and only if $D(n,S)$ is integral over F.

Let $H' = Gal(\mathbb{Q}(\zeta_n)/F), H = Gal(F \cdot \mathbb{Q}(\zeta_g)/F),\ G' = Gal(\mathbb{Q}(\zeta_g)/F \bigcap \mathbb{Q}(\zeta_g))$, and $G = Gal(\mathbb{Q}(\zeta_g)/\mathbb{Q})$. By Galois theory(see [L], P.266), $H \cong G' \subseteq G$. Denote π the translation operation of G' on G and $f : G \longrightarrow (\mathbb{Z}/g\mathbb{Z})^*$ the map defined by $f : \sigma \mapsto a$ with $\sigma(\zeta_g) = \zeta_g^a$. The map f is an isomorphism of groups, by which π also can be considered as an operation on $(\mathbb{Z}/g\mathbb{Z})^*$. For $G_n(p) = p(\mathbb{Z}/g\mathbb{Z})^*$, we can define an operation π' of G' on $G_n(p)$ by $\pi'_g(x) = p\pi_g(x')$ with $x = px' \in G_n(p), g \in G'$ and $x' \in (\mathbb{Z}/g\mathbb{Z})^*$. It is easy to see the number r_p of all orbits under π' is equal to $[F \bigcap \mathbb{Q}(\zeta_g) : \mathbb{Q}]$ and each orbit $M_i(1 \leq i \leq r_p)$ has $[\mathbb{Q}(\zeta_g) : F \bigcap \mathbb{Q}(\zeta_g)]$ elements. By Galois theory, H', when restricted to $F \cdot \mathbb{Q}(\zeta_g)$, is equal to H and $H \cong G'$. Hence π' can be regarded as an operation of H' on $G_n(p)$, under which there are the same r_p orbits. So $G_n(p)$ is the disjoint union of the distinct M_i's and we can write $G_n(p) = \bigsqcup_{i=1}^{r_p} M_i$. It is clear that $\{1, 2, \cdots, n-1\}$ has a partition written as $\{1, 2, \cdots, n-1\} = \bigsqcup_{p \in d(n)} G_n(p)$. So $\{1, 2, \cdots, n-1\}$ has a new partition $\bigsqcup_{i=1}^{r(n,K)} M_i$, where M_i is an orbit for some $p \in d(n)$ and $r(n,K) = \sum_{p \in d(n)} r_p$.

3 Proofs of Main Results

If S is a union of some M_i's, we have $\sigma S = S$ for each $\sigma \in H'$ by the construction of M_i's. So $\sigma(\lambda_r) = \lambda_r(0 \leq r \leq n-1)$, which shows that $\lambda_r \in F \subseteq K$(see [L], P.262) and $D(n,S)$ is integral over K. We have the following theorem.

Theorem 1. The circulant directed graph $D(n, S)$ is integral over K if and only if S is a union of some M_i's.

For the discussion above, it suffices to prove the necessity. Before proving, we need the following lemma.

Denote v_i the $(n-1)-$dimension vector corresponding to the orbit M_i with 1 at $j-$th entry for all $j \in M_i$ and 0 otherwise and M the set $\{v_i | 1 \leq i \leq r(n, K)\}$. Let $\Gamma = (\gamma_{st})$ be an $(n-1)-$order square matrix defined by $\gamma_{st} = \zeta_n^{st}$. Notice that Γ is invertible and $\Gamma v_i \in F^{n-1}(1 \leq i \leq r(n, K))$. Let $V = \{v \in \mathbb{Q}^{n-1} | \Gamma v \in F^{n-1}\}$ and W be the vector space over $\mathbb{Q}$ spanned by the vectors belongs to M. We have the following results.

Lemma 1. We have $V = W$ and M is an orthogonal basis of V.

Proof. Firstly, we claim that if $v \in V$, then $\Gamma v \in W \bigotimes_{\mathbb{Q}} F$. Let $v = (w_1, w_1, \cdots, w_{n-1})^T \in V$ and $u = \Gamma v = (u_1, u_1, \cdots, u_{n-1})^T \in F^{n-1}$. It is enough to show $u_k = u_l$ if k, l in the same M_i. Recall that M_i is an orbit obtained from $G_n(p)$ for some $p \in d(n)$ and $n = pg$. Let $f(x) = u_k - \sum_{h=1}^{n-1} w_h x^h$ be a polynomial in $F[x]$. So $f(\zeta_n^k) = 0$ for $u_k = \sum_{h=1}^{n-1} w_h \zeta_n^{kh}$. Because $k, l \in M_i, \zeta_n^k$ and ζ_n^l are two primitive $g-$th roots of unity. So ζ_n^k and $\zeta_n^l \in \mathbb{Q}(\zeta_g)$, which shows that $f(x) \in (F \bigcap \mathbb{Q}(\zeta_g))[x]$. By the construction of M_i, there exists an element $\sigma \in Gal(\mathbb{Q}(\zeta_g)/F \bigcap \mathbb{Q}(\zeta_g))$ such that $\sigma(\zeta_n^k) = \zeta_n^l$, which means that ζ_n^k is a conjugate element of ζ_n^l over $F \bigcap \mathbb{Q}(\zeta_g)$ in $\mathbb{Q}(\zeta_g)$. Hence ζ_n^k and ζ_n^l have the same minimal polynomial over $F \bigcap \mathbb{Q}(\zeta_g)$, which divides $f(x)$. So $f(\zeta_n^l) = 0$, that is $u_l = \sum_{h=1}^{n-1} w_h \zeta_n^{lh} = u_k$. Using the result above, it is easy to get $\Gamma(V \bigotimes_{\mathbb{Q}} F) \subseteq W \bigotimes_{\mathbb{Q}} F$, which shows that $V \bigotimes_{\mathbb{Q}} F = W \bigotimes_{\mathbb{Q}} F$.

Therefore $V = W$ and M is an orthogonal basis of V. The proof is completed. $\square$

Now it comes to prove the Theorem 1.

Proof. Consider the vector $v \in \mathbb{Q}^{n-1}$ such that 1 at $j-$th entry for all $j \in S$ and 0 otherwise. Since $D(n,S)$ is a circulant digraph integral over $K, \Gamma v = (\lambda_1, \lambda_2, \cdots, \lambda_{n-1})^T \in F^{n-1}$. Hence $v \in V$ and by Lemma1 $v = \sum_{i=1}^{r(n,K)} c_i v_i$ for some rational coefficients c_i's. By the construction of M and v, S is a union of those M_i's with $c_i = 1$. The proof is completed. $\square$

By the Theorem 1, we obtain the following two corollaries.

Given a fixed pair (n,K), we have the partition $\{1,2,\cdots,n-1\} = \bigsqcup_{i=1}^{r(n,K)} M_i$, where $r(n,K) = \sum_{p \in d(n)} [K \bigcap \mathbb{Q}(\zeta_{\frac{n}{p}}) : \mathbb{Q}]$. Therefore there are $2^{r(n,K)}$ distinct S for circulant digraph integral over K.

Corollary 1. There are at most $2^{r(n,K)}$ basic circulant digraphs integral over K on n vertices.

Replacing K by the specific quadratic field $\mathbb{Q}(i)$ and let $n = 2^t k (t > 2)$, it is easy to check that the partition of $\{1,2,\cdots,n-1\}$ described in Part 3 of paper [XM] is the same as $\bigsqcup_{i=1}^{r(n,K)} M_i$. By Theorem 1, we infer that the Conjecture 3.3 in [XM] is affirmative. Using the same notations of [XM], we have the following corollary.

Corollary 2. (Conjecture 3.3 in [XM]) Let $D(\mathbb{Z}_n, S)$ be a circulant digraph on $n = 2^t k (t > 2)$ vertices, then $D(\mathbb{Z}_n, S)$ is Gauss integral if and only if S is a union of the M's.

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